San Diego Math League Gauss Olympiad
March 28, 2009

1. Find all ordered pairs of positive integers (a, b) such that

$$\frac{1}{a} + \frac{a}{b} + \frac{1}{ab} = 1.$$

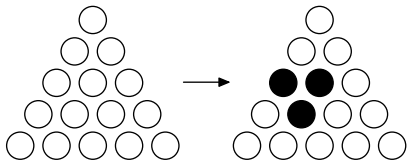
2. In triangle ABC, $AB = 20$, $AC = 21$, and $BC = 29$. The points D and E lie on the line segment $\overline{BC}$, with $BD = 8$ and $EC = 9$. Find $\angle DAE$.

3. Find all real numbers x such that

$$\sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}} = 2.$$

4. Prove that the product of three consecutive positive integers cannot be a perfect square.

5. Let $n \geq 2$ be a positive integer. We are given $n(n+1)/2$ lights, arranged in a triangular array. All the lights are initially turned on. A move consists of selecting three lights that are mutually adjacent, and flipping them, so that the lights that are on get turned off, and the lights that are off get turned on. A sample move is shown below.



Find all positive integers n for which all the lights can be turned off by a sequence of moves.